ON THE SYSTEM OF DIFFERENTIAL EQUATIONS OF EQUILIBRIUM OF SHELLS OF REVOLUTION UNDER BENDING LOADS

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Bending or wind pressure loading is called a loading which varies with the angle ϕ (see figure) according to the formulas

$$q_1 = q_{11}\cos\varphi, \qquad q_2 = q_{21}\sin\varphi, \qquad q_n = q_{n1}\cos\varphi \tag{1}$$

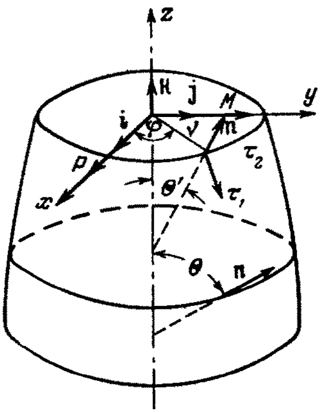
All stress resultants, as well as the displacements in the shell, vary with ϕ in conformity with the same law:

$$\begin{gathered} T_1 = t_1\cos\varphi, \qquad T_2 = t_2\cos\varphi, \qquad S = s\sin\varphi \\ M_1 = m_1\cos\varphi, \quad M_2 = m_2\cos\varphi, \quad H = h_1\sin\varphi \\ u = u_1\cos\varphi, \qquad v = v_1\sin\varphi, \qquad w = w_1\cos\varphi \\ N_1 = n_1\cos\varphi; \quad \varepsilon_1 = \varepsilon_{11}\cos\varphi, \quad \varepsilon_2 = \varepsilon_{21}\cos\varphi, \quad \omega = \omega_1\sin\varphi \\ N_2 = n_2\sin\varphi, \quad \varkappa_1 = \varkappa_{11}\cos\varphi, \quad \varkappa_2 = \varkappa_{21}\cos\varphi, \quad \tau = \tau_1\sin\varphi \end{gathered} \tag{2}$$

Therefore, in the case of the loading just indicated, the system of differential equations of a shell of revolution becomes a system of ordinary differential equations, but this system is of the eighth order, as in the general case of loading.

The possibility of lowering the order of the system was first discovered by Schwerin in the case of the spherical shell. It is also known that the analysis of a cylindrical shell under bending loads can be reduced to the treatment of a differential equation of the fourth order with constant coefficients [1].

Novozhilov has reduced the problem of analyzing shells having the shape of any surface of revolution, under wind pressure loading, to that of one differential equation of the second order with respect to an unknown complex function. To this end he introduced a complex transformation, of the



fundamental equations of the theory of shells. In this way the order of the original system in terms of real quantities becomes halved. In carrying out Novozhilov's complex transformation the original system of differential equations can be simplified, however, in one of the following ways. (1) Poisson's ratio μ is assumed to be zero; (2) If $\mu \neq 0$, a number of terms is omitted in the compatibility equations (formula (16.5), p. 71, [3]).

The present note shows that the order of the system can be lowered from eighth to fourth with no changes at all in the original system.

As such a system we use the five equations of equilibrium of shells ((1.5) [2]), the three equations of continuity (5.1), p. 29 [3]) and the six elasticity relations ((12.1), p. 56 [3]).

After differentiation with respect to the coordinate ϕ, the equations of statics ((1.5) [2]) assume the form

$$\frac{d}{d\theta}(t_1R_2\sin\theta)+R_1\left(s+\frac{h_1}{R_2}\right)-t_2R_1\cos\theta+n_1R_2\sin\theta+q_{11}R_1R_2\sin\theta=0$$

$$\begin{gathered}\frac{d}{d\theta}\left[R_2\sin\theta\left(s+\frac{h_1}{R_2}\right)\right]+R_1\cos\theta\left(s+\frac{h_1}{R_1}\right)- \\ -t_2R_1+n_2R_1\sin\theta+q_{21}R_1R_2\sin\theta=0\end{gathered} \tag{3}$$

$$\frac{1}{R_1R_2\sin\theta}\left[\frac{d}{d\theta}(n_1R_2\sin\theta)+R_1n_2\right]-\frac{t_1}{R_1}-\frac{t_2}{R_2}+q_{n1}=0$$

$$n_1 = \frac{1}{R_1 R_2 \sin\theta}\left[\frac{d}{d\theta}(m_1 R_2 \sin\theta) + R_1 h_1 - R_1 \cos\theta m_2\right]$$
$$n_2 = \frac{1}{R_1 R_2 \sin\theta}\left[\frac{d}{d\theta}(h_1 R_2 \sin\theta) - R_1 m_2 + R_1 \cos\theta h_1\right] \tag{4}$$

where R_1 and R_2 are the principal radii of curvature of the middle surface of the shell. Multiplying the first and the third of (3) by $-\cos\theta$ and $-\sin\theta$, respectively, and adding the reults to the second, we find

$$-\frac{d}{d\theta}(t_1 R_2 \sin\theta\cos\theta) - \frac{d}{d\theta}(n_1 R_2 \sin^2\theta) + \frac{d}{d\theta}\left[R_2 \sin\theta\left(s + \frac{h_1}{R_2}\right)\right] +$$
$$+ (-q_{11}\cos\theta + q_{21} - q_{n1}\sin\theta) R_1 R_2 \sin\theta = 0 \tag{5}$$

Integration of (5) leads to the first integral of system (3):

$$-t_1 R_2 \sin\theta\cos\theta - n_1 R_2 \sin^2\theta + R_2 \sin\theta\left(s + \frac{h_1}{R_2}\right) +$$
$$+ \int_{\theta'}^{\theta} (-q_{11}\cos\theta + q_{21} - q_{n1}\sin\theta) R_1 R_2 \sin\theta d\theta = C_1 \tag{6}$$

The elimination of the quantities m_2 and n_2 from (4) and the third of (3), allowing for (6), gives another integral of the equations of statics:

$$n_1 R_2^2 \sin^2\theta\cos\theta + h_1 R_2 \sin\theta\cos\theta - t_1 R_2^2 \sin^3\theta - m_1 R_2 \sin\theta +$$
$$+ \int_{\theta'}^{\theta} R_1 \sin\theta\left[\int_{\theta'}^{\theta} (-q_{11}\cos\theta + q_{21} - q_{n1}\sin\theta) R_1 R_2 \sin\theta d\theta\right] d\theta +$$
$$+ \int_{\theta'}^{\theta} (q_{n1}\cos\theta - q_{11}\sin\theta) R_1 R_2^2 \sin^2\theta d\theta = C_2 + C_1 \int_{\theta'}^{\theta} R_1 \sin\theta d\theta \tag{7}$$

The integrals (6) and (7) represent the conditions of equilibrium of the end element of the shell enclosed between the two parallel sections θ' and θ. Stating these conditions directly, we find the constants C_1 and C_2.

The system of stresses in the section θ = const is statically equivalent to the stress resultant $\mathbf{K}_1 \nu\, d\phi$ and the moment $\mathbf{M}_1 \nu\, d\phi$ ($\nu = R_2 \sin\theta$):

$$\mathbf{K}_1 = T_1 \boldsymbol{\tau}_1 + T_{12} \boldsymbol{\tau}_2 + Q_1 \mathbf{n} = (T_1 \cos\theta\cos\varphi - T_{12}\sin\varphi + Q_1 \sin\theta\cos\varphi)\,\mathbf{i} +$$
$$+ (T_1 \cos\theta\sin\varphi + T_{12}\cos Q + Q_1 \sin\theta\sin\varphi)\,\mathbf{j} + (-T_1 \sin\theta + Q_1\cos\theta)\,\mathbf{k}$$

$$\mathbf{M}_1 = M_1\boldsymbol{\tau}_2 - H_{12}\boldsymbol{\tau}_1 = (-H_{12}\cos\theta\cos\varphi + M_1\sin\varphi)\,\mathbf{i} + \\ + (-H_{12}\cos\theta\sin\varphi + M_1\cos\varphi)\,\mathbf{j} + (-H_{12}\sin\theta + M_1\cos\theta)\,\mathbf{k} \tag{8}$$

The shear force Q_1, the stress resultants T_{12}, T_{21} and the twisting moments H_{12}, H_{21} are connected with the quantities S, H and N by relations of the form [2]

$$S = T_{12} - \frac{H_{21}}{R_2} = T_{21} - \frac{H_{12}}{R_1}, \qquad 2H = H_{21} + H_{12} \tag{9}$$

$$Q_1 = N_1 + \frac{1}{R_2\sin\theta}\frac{\partial}{\partial\varphi}\left(\frac{H_{21} - H_{12}}{2}\right)$$

Equating to zero the resultant force and the resultant moment of external and internal forces at the section θ = const, we get

$$\int_0^{2\pi} \boldsymbol{K}_1\nu d\varphi + \boldsymbol{P}\boldsymbol{i} + \int_{\theta'}^{\theta}\int_0^{2\pi} (q_1\boldsymbol{\tau}_1 + q_2\boldsymbol{\tau}_2 + q_n\boldsymbol{n})\,R_1R_2\sin\theta\,d\theta d\varphi = 0 \tag{10}$$

$$\int_0^{2\pi} (\boldsymbol{M}_1\nu + \boldsymbol{r}_0\times\nu\boldsymbol{K}_1)\,d\varphi + \int_{\theta'}^{\theta}\int_0^{2\pi} \boldsymbol{r}\times(q_1\boldsymbol{\tau}_1 + q_2\boldsymbol{\tau}_2 + q_n\boldsymbol{n})\,R_1R_2\sin\theta\,d\theta d\varphi + \\ + \left(M - P\int_0^{\theta} R_1\sin\theta\,d\theta\right)\boldsymbol{j} = 0 \tag{11}$$

$$\boldsymbol{r}_0 = x\boldsymbol{i} + y\boldsymbol{j}, \qquad x = R_2\sin\theta\cos\varphi, \qquad y = R_2\sin\theta\sin\varphi \tag{12}$$

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} - \boldsymbol{k}\int_{\theta'}^{\theta} R_1\sin\theta d\theta$$

Integrating (10) and (11) with respect ot ϕ, and allowing for (1), (2), (8), (9), we find

$$t_1R_2\sin\theta\cos\theta - sR_2\sin\theta - h_1\sin\theta + n_1R_2\sin^2\theta + \\ + \int_{\theta'}^{\theta} (q_{11}\cos\theta - q_{21} + q_{n_1}\sin\theta)\,R_1R_2\sin\theta d\theta = -\frac{P}{\pi} \tag{13}$$

$$-m_1R_2\sin\theta - t_1R_2^2\sin^3\theta + n_1R_2^2\sin^2\theta\cos\theta + h_1R_2\sin\theta\cos\theta + \\ + \int_{\theta'}^{\theta} (-q_{11}\sin\theta + q_{n1}\cos\theta)\,R_1R_2^2\sin^2\theta d\theta -$$

$$\int_{\theta'}^{\theta} (q_{11}\cos\theta - q_{21} + q_{n1}\sin\theta) R_1 R_2 \sin\theta \left[\int_{\theta'}^{\theta} R_1 \sin\theta\, d\theta\right] d\theta =$$

$$= \frac{M}{\pi} + \frac{P}{\pi}\int_{\theta'}^{\theta} R_1 \sin\theta\, d\theta \tag{14}$$

Comparison of (6), (7) with (13), (14) shows that

$$C_1 = \frac{P}{\pi}, \qquad C_2 = \frac{M}{\pi} \tag{15}$$

i.e. the constants C_1 and C_2 of integration are proportional to the force and the moment, respectively, applied at the end section of the shell.

Having derived two integrals of the equations of statics, we obtain without difficulties two integrals of the continuity equations as well. To this end it is only necessary to make use of the statico-geometric analogy, that is, in the case under consideration, of the circumstance that the continuity equations in terms of the strain components ((5.1), p. 29, and (8.2), p. 39, [3]) and the homogeneous equations of statics, after the elimination of N_1 and N_2, contain the same differential operators, while the quantities

$$\begin{array}{lll} T_1,\ \varkappa_2; & M_1,\ (-\varepsilon_2); & S\ \ (-\tau) \\ T_2,\ \varkappa_1; & M_2,\ (-\varepsilon_1); & H,\ (^1/_2\omega) \end{array} \tag{16}$$

appear in these equations in the same manner.

Transforming (6) and (7), allowing for (4), putting $q_{11} = q_{21} = q_{n1} = 0$, and replacing the variables $(t_1,\ t_2,\ \ldots)$ by $(\kappa_{21},\ \kappa_{11},\ \ldots$ etc.) in conformity with (16), we get

$$R_2 \sin\theta\,(\varkappa_{21}\cos\theta + \tau_1) - \frac{R_2}{R_1}\sin^2\theta\,\frac{d\varepsilon_{21}}{d\theta} - (\varepsilon_{21} - \varepsilon_{11})\sin\theta\cos\theta + C_3 = 0 \tag{17}$$

$$-\frac{R_2\sin\theta\cos\theta}{R_1}\frac{d\varepsilon_{21}}{d\theta} + \omega\cos\theta + \varepsilon_{21}\sin^2\theta + \varepsilon_{11}\cos^2\theta - \varkappa_{21}R_2\sin^2\theta =$$

$$= \frac{1}{R_2\sin\theta}\left(C_4 + C_3\int_{\theta'}^{\theta} R_1\sin\theta\, d\theta\right) \tag{18}$$

The strain components in terms of displacements are given by the formulas

$$\varepsilon_{11} = \frac{1}{R_1}\frac{du_1}{d\theta} + \frac{w_1}{R_1}, \qquad \varepsilon_{21} = \frac{w_1\sin\theta + v_1 + u_1\cos\theta}{R_2\sin\theta}$$

$$
\begin{aligned}
\omega_1 &= \frac{1}{R_1}\frac{dv_1}{d\theta} - \frac{u_1 + v_1\cos\theta}{R_2\sin\theta} \\
\varkappa_{11} &= \frac{-1}{R_1}\frac{d}{d\theta}\left(\frac{1}{R_1}\frac{dw_1}{d\theta} - \frac{u_1}{R_1}\right) \\
\varkappa_{21} &= \frac{1}{R_2\sin\theta}\left[\frac{w_1 + v_1\sin\theta}{R_2\sin\theta} - \frac{\cos\theta}{R_1}\left(\frac{dw_1}{d\theta} - u_1\right)\right] \\
\tau_1 &= \frac{1}{R_2\sin\theta}\left[\frac{1}{R}\frac{dw_1}{d\theta} - \frac{u_1}{R_1} - \frac{\cos\theta\,(w_1 + v_1\sin\theta)}{R_2\sin\theta} + \frac{\sin\theta}{R_1}\frac{dv_1}{d\theta}\right]
\end{aligned}
\tag{19}
$$

Substituting these expressions into the equations (17), (18) and considering that this substitution must satisfy these two equations identically, we find

$$C_3 = C_4 = 0$$

As a result of the determination of four integrals of the original system of differential equations, the order of this system is halved. To show this explicitly we write down the system arrived at:

$$
\begin{aligned}
&\nu t_1 - \nu s\cos\theta - 2h_1\sin\theta\cos\theta + m_1\sin\theta = \\
&\quad = \nu\left[f_0(q_{11}, q_{21}, q_{n1}) + f_1(P, M)\right] + \cos\theta\int_{\theta'}^{\theta} q_{21}R_1R_2\sin\theta\, d\theta
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
&\frac{1}{R_1\nu}\left[\frac{d}{d\theta}(m_1\nu) + R_1h_1 - m_2R_1\cos\theta\right] - s\sin\theta + \frac{h_1}{\nu}(\cos^2\theta - \sin^2\theta) - \\
&\quad - \frac{m_1\cos\theta}{\nu} = F_0(q_{11}, q_{21}, q_{n1}) + F_1(P, M) + \frac{\sin\theta}{\nu}\int_{\theta'}^{\theta} q_{21}R_1R_2\sin\theta\, d\theta
\end{aligned}
\tag{21}
$$

$$\nu\varkappa_{21} + \nu\tau_1\cos\theta - \varepsilon_{21}\sin\theta - \omega\sin\theta\cos\theta = 0 \tag{22}$$

$$\frac{1}{R_1}\frac{d\varepsilon_{21}}{d\theta} - \tau_1\sin\theta - \frac{\omega\cos^2\theta}{\nu} - \frac{\varepsilon_{11}\cos\theta}{\nu} = 0 \tag{23}$$

$$
\begin{aligned}
&\nu\frac{ds}{d\theta} + 2sR_1\cos\theta + 2\frac{dh_1}{d\theta}\sin\theta + 2h_1\cos\theta - R_1t_2 + \\
&\quad + 2h_1\frac{R_1\sin\theta\cos\theta}{\nu} - m_2\frac{R_1\sin\theta}{\nu} + q_{21}R_1R_2\sin\theta = 0
\end{aligned}
\tag{24}
$$

$$-R_1\varkappa_{11} - \nu\frac{d\tau}{d\vartheta} - 2R_1\cos\theta\tau + \omega\cos\theta + \sin\theta\frac{d\omega}{d\vartheta} + \tag{25}$$
$$+ \frac{R_1\sin\theta\cos\theta}{\nu}\omega + \frac{R_1\sin\theta}{\nu}\varepsilon_{11} = 0$$

$$f_0(q_{11}, q_{21}, q_{n1}) = f_0(q) = -\frac{\cos\theta}{\nu}\int\limits_{\theta'}^{\theta}(q_{11}\cos\theta + q_{n1}\sin\theta)R_1R_2\sin\theta\, d\theta +$$
$$+ \frac{\sin\theta}{\nu^2}\left\{\int\limits_{\theta'}^{\theta}(-q_{11}\sin\theta + q_{n1}\cos\theta)R_1R_2^2\sin^2\theta\, d\theta -\right.$$
$$\left.-\int\limits_{\theta'}^{\theta}(q_{11}\cos\theta - q_{21} + q_{n1}\sin\theta)R_1R_2\sin\theta\left[\int\limits_{\theta'}^{\theta}R_1\sin\theta\, d\theta\right]d\theta\right\}$$

$$f_1(P, M) = -\frac{P\cos\theta}{\pi\nu} - \left(\frac{M}{\pi} + \frac{P}{\pi}\int\limits_{\theta'}^{\theta}R_1\sin\theta\, d\theta\right)\frac{\sin\theta}{\nu^2} \tag{26}$$

$$F_0(q_{11}, q_{21}, q_{n1}) = F_0(q) = -\frac{\sin\theta}{\nu}\int\limits_{\theta'}^{\theta}(q_{11}\cos\theta + q_{n1}\sin\theta)R_1R_2\sin\theta\, d\theta -$$
$$-\frac{\cos\theta}{\nu^2}\left\{\int\limits_{\theta'}^{\theta}(-q_{11}\sin\theta + q_{n1}\cos\theta)R_1R_2^2\sin^2\theta\, d\theta -\right.$$
$$\left.-\int\limits_{\theta'}^{\theta}(q_{11}\cos\theta - q_{21} + q_{n1}\sin\theta)R_1R_2\sin\theta\left[\int\limits_{\theta'}^{\theta}R_1\sin\theta\, d\theta\right]d\theta\right\}$$

$$F_1(P, M) = -\frac{P\sin\theta}{\pi\nu} + \left(\frac{M}{\pi} + \frac{P}{\pi}\int\limits_{\theta'}^{\theta}R_1\sin\theta\, d\theta\right)\frac{\cos\theta}{\nu^2}$$

Equation (20) is derived from the integrals (13) and (14) by eliminating the quantity n_1; (21) is obtained from the same integrals by the elimination of t_1; (22) and (23) are continuity equations derived from the integrals (17) and (18) by eliminating the quantities $d\epsilon_{21}/d\theta$ and κ_{21}, respectively; (24) is the second equation of statics obtained by eliminating n_2; (25) is the second continuity equation. We note that the compatibility equations (22), (23), (25) can be expressed, with the aid of elasticity relations, in terms of stress resultants; this would lead to a system of six equations sufficient to determine the six unknowns t_1, t_2, s, m_1, m_2, h_1. This system would, however, be a very complicated one in spite of the lowering of the order of the differential equations. To reduce it to a simpler form we use a procedure similar to that introduced by Meissner for transforming the system of equations of the axisymmetrical shell problem.

From relations (19) we immediately derive

$$\varkappa_{11} = \frac{1}{R_1}\frac{d\psi}{d\theta} + \frac{\psi\cos\theta}{\nu} + \frac{\varepsilon_{21}\sin\theta}{\nu}, \qquad \tau_1 = -\frac{\psi}{\nu} + \frac{\omega_1\sin\theta}{\nu}$$

$$\varkappa_{21} = \frac{\psi\cos\theta}{\nu} + \frac{\varepsilon_{21}\sin\theta}{\nu} \tag{27}$$

where

$$\psi = -\frac{1}{R_1}\left(\frac{dw_1}{d\theta} - u_1\right) + \frac{w_1\cos\theta - u_1\sin\theta}{\nu} \tag{28}$$

It is not difficult to verify that eliminating ψ from the three equations (27) leads to two relations identical with (22) and (25). Thus, with the aid of (27), one of the integrals and the second compatibility equation are identically satisfied.

Similarly, we represent the stress resultants by means of a certain function in connection with loading terms selected in such a manner as to satisfy the two nonhomogeneous equations of statics:

$$t_2 = -\frac{1}{R_1}\frac{dV}{d\theta} + \frac{V\cos\theta}{\nu} - \frac{m_2\sin\theta}{\nu} - \frac{\cos\theta}{\nu}\int_{\theta'}^{\theta} q_{21}R_1R_2\sin\theta\, d\theta$$

$$t_1 = \frac{V\cos\theta}{\nu} - \frac{m_1\sin\theta}{\nu} + f_0(q) + f_1(P, M) \tag{29}$$

$$s = \frac{V}{\nu} - \frac{2h_1}{\nu}\sin\theta - \frac{1}{\nu}\int_{\theta'}^{\theta} q_{21}R_1R_2\sin\theta\, d\theta$$

The elimination of function V from (29) actually leads to two equations identical with (20) and (24).

To obtain equations for determinating the functions ψ and V, it remains to make use of the equations (21) and (23) and the elasticity relations, which permits all stress resultants to be expressed in terms of ψ and V.

Indeed, from (27), (29) and from the elasticity relations ((12.1), [3]), disregarding quantities of the order of magnitude of h^2/ν^2 as compared with the unity, we obtain

$$\frac{1}{D}m_1 = \frac{1}{R_1}\frac{d\psi}{d\theta} + \frac{(1+\mu)\psi\cos\theta}{\nu} + \frac{1-\mu^2}{Eh}\frac{V\sin\theta\cos\theta}{\nu^2} +$$

$$+ \frac{\sin\theta}{Eh\nu}(1-\mu^2)[f_0(q) + f_1(P, M)]$$

$$\frac{1}{D} m_2 = \frac{\mu}{R_1}\frac{d\psi}{d\theta} + \frac{(1+\mu)\,\psi\cos\theta}{\nu} + \frac{\sin\theta\,(1-\mu^2)}{Eh\nu}\left[\frac{V\cos\theta}{\nu} + \frac{1}{R_1}\frac{dV}{d\theta}\right] -$$

$$- \frac{(1-\mu^2)\sin\theta\cos\theta}{Eh\nu^2}\int_{\theta'}^{\theta} q_{21}R_1R_2\sin\theta\,d\theta,$$

$$\frac{h_1}{D} = -\frac{\psi(1-\mu)}{\nu} + \frac{2(1-\mu^2)}{Eh}\frac{V\sin\theta}{\nu^2} - \frac{2(1-\mu^2)\sin\theta}{Eh\nu^2}\int_{\theta'}^{\theta} q_{21}R_1R_2\sin\theta\,d\theta$$

$$t_1 = \frac{V\cos\theta}{\nu} - \frac{\sin\theta D}{\nu}\left[\frac{1}{R_1}\frac{d\psi}{d\theta} + \frac{(1+\mu)\,\psi\cos\theta}{\nu}\right] + f_0(q) + f_1(P, M) \tag{30}$$

$$t_2 = \frac{1}{R_1}\frac{dV}{d\theta} + \frac{V\cos\theta}{\nu} - \frac{\sin\theta D}{\nu}\left[\frac{\mu}{R_1}\frac{d\psi}{d\theta} + \frac{(1+\mu)\,\psi\cos\theta}{\nu}\right] -$$

$$- \frac{-\cos\theta}{\nu}\int_{\theta'}^{\theta} q_{21}R_1R_2\sin\theta\,d\theta$$

$$s = \frac{V}{\nu} + \frac{2(1-\mu)\sin\theta D}{\nu}\frac{\psi}{\nu} - \frac{1}{\nu}\int_{\theta'}^{\theta} q_{21}R_1R_2\sin\theta\,d\theta$$

where

$$D = \frac{Eh^3}{12(1-\mu^2)}$$

Substituting the expressions for the stress resultants, that is formulas (30), into (21), and into (23) expressed in terms of stress resultants, i.e.

$$\frac{1}{R_1}\frac{d}{d\theta}(t_2 - \mu t_1) - \frac{12(1+\mu)}{h^2}h_1\sin\theta - \frac{2(1+\mu)\,s\cos^2\theta}{\nu} - (t_1 - \mu t_2)\frac{\cos\theta}{\nu} = 0 \tag{31}$$

for the functions ψ and V we obtain two equations:

$$\frac{D}{R_1^2}\frac{d^2\psi}{d\theta^2} + \frac{D}{R_1\nu}\frac{d\psi}{d\theta}\left[-\frac{\nu}{R_1^2}\frac{dR_1}{d\theta} + \cos\theta\right] + \frac{D\psi}{R_1\nu}\left[-(1+\mu)\sin\theta - \right.$$

$$\left. - \frac{2(1-\mu)R_1}{\nu} - \frac{2(1+\mu)R_1\cos^2\theta}{\nu}\right] + V\left[-\frac{\sin\theta}{\nu} + \frac{\sin\theta}{\nu}\frac{h^2}{12\nu^2}\cos^2\theta + \right.$$

$$\left. + \frac{1}{R_1}\frac{h^2}{12\nu^2}(\cos^2\theta - \sin^2\theta)\right] = F_0(q) + F_1(P, M) \tag{32}$$

$$\frac{1}{R_1^2}\frac{d^2V}{d\theta^2}+\frac{dV}{d\theta}\left(-\frac{1}{R_1^3}\frac{dR_1}{d\theta}+\frac{\cos\theta}{R_1\nu}\right)+V\left[-\frac{(1-\mu)\sin\theta}{R_1\nu}-\frac{2(1-\mu)}{\nu^2}-\right.$$

$$\left.-\frac{2(1-\mu)\cos^2\theta}{\nu^2}\right]+(1-\mu^2)D\psi\left[\frac{12}{h^2}\frac{\sin\theta}{\nu}-\frac{\sin\theta\cos^2\theta}{\nu^3}+\frac{\sin^2\theta-\cos^2\theta}{R_1\nu^2}\right]=$$

$$=\frac{\mu}{R_1}\frac{d[f_0(q)+f_1(P,M)]}{d\theta}+\frac{\cos\theta}{\nu}[f_0(q)+f_1(P,M)]+f(q_{21}) \qquad (33)$$

where

$$f(q_{21})=\frac{1}{R_1}\frac{d}{d\theta}\left[\frac{\cos\theta}{\nu}\int_{\theta'}^{\theta}q_{21}R_1R_2\sin\theta\,d\theta\right]+$$

$$+[\mu\cos^2\theta-2(1+\mu)]\frac{1}{\nu^2}\int_{\theta'}^{\theta}q_{21}R_1R_2\sin\theta\,d\theta$$

We now introduce new functions v and ψ_1, connected with V and ψ by the relations

$$v=\frac{V}{R_1^2},\qquad \psi_1=\frac{Eh\psi}{R_1} \qquad (34)$$

and subsequently we use the notation

$$\frac{12(1-\mu^2)R_1^2}{h^2}=4\gamma^4$$

Multiplying (32) by $12R_1(1-\mu^2)/h^2$ and (33) by $(-2i\gamma^2)$, and adding the results term by term, we obtain one equation for the complex function $\sigma=\psi_1-2i\gamma^2 v$, namely

$$\frac{d^2\sigma}{d\theta^2}+\frac{d\sigma}{d\theta}\left(-\frac{1}{R_1}\frac{dR_1}{d\theta}+\frac{R_1\cos\theta}{\nu}\right)+\sigma\left(-\frac{R_1\sin\theta}{\nu}-\frac{2R_1^2}{\nu^2}-\right.$$

$$\left.-\frac{2R_1^2\cos^2\theta}{\nu^2}\right)+\bar{\sigma}\mu\left(-\frac{R_1\sin\theta}{\nu}+\frac{2R_1^2}{\nu^2}\sin^2\theta\right)+ \qquad (35)$$

$$+2i\gamma^2\left[-\frac{R_1\sin\theta}{\nu}+\frac{R_1\sin\theta}{\nu}\frac{h^2}{12\nu^2}\cos^2\theta+\frac{h^2}{12\nu^2}(1-2\sin^2\theta)\right]\sigma=\Phi(q,P,M)$$

where

$$\Phi(q,P,M)=[F_0(q)+F_1(P,M)]\frac{12(1-\mu^2)R_1}{h^2}-$$

$$-2i\gamma^2\left[\frac{\mu}{R_1}\frac{d(f_0+f_1)}{d\theta}+\frac{\cos\theta}{\nu}(f_0+f_1)+f(q_{21})\right] \qquad (36)$$

To remain within the accuracy limits of the theory of thin shells, in the coefficient of the unknown function σ in (35), we omit terms of the order of magnitude of h/R_1 and of h^2/ν^2 as compared with the unity. This ultimately leads to

$$\frac{d^2\sigma}{d\theta^2} + \frac{d\sigma}{d\theta}\left(-\frac{1}{R_1}\frac{dR_1}{d\theta} + \frac{R_1\cos\theta}{\nu}\right) - 2i\gamma^2\frac{R_1\sin\theta}{\nu}\sigma = \Phi(q, P, M) \qquad (37)$$

BIBLIOGRAPHY

1. Koshutin, M.P., Zadachi izgiba tsilindricheskoi obolochki (Problems of Bending of Cylindrical Shells). *Trudy LPI* No. 192, Mashgiz, 1958.

2. Lur'e, A.I., Ob uravneniiakh obshchei teorii uprugikh obolochek (On the Equations of the General Theory of Elastic Shells). *PMM* Vol. 14, No. 5, 1950.

3. Novozhilov, V.V., *Teoriia tonkikh obolochek* (*Theory of Thin Shells*). Sudpromgiz, 1951.

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